DEGREES OF EXCEPTIONAL CHARACTERS
OF CERTAIN FINITE GROUPS

BY

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ABSTRACT. Let G be a finite group whose order is divisible by a prime p to the first power only. Restrictions beyond the known congruences modulo p are shown to hold for the degrees of the exceptional characters of G, under the assumptions that either all p'-elements centralizing a Sylow p-subgroup are in fact central in G and there are at least three conjugacy classes of elements of order p, or that the characters in question lie in the principal p-block. Results of Feit and the author are thereby generalized.

1. Introduction. This paper proves two theorems on the possible degrees of irreducible characters of some finite groups whose orders are divisible by a prime to the first power only. We first set some notation and hypotheses, and recall a few basic results.

G denotes a finite group, p an odd prime, P a Sylow p-subgroup of G. N and C are the normalizer, resp. centralizer, of P in G, $e = |N : C|$ and $t = (p - 1)/e$. Z is the center of G. We follow the terminology of [6], [1] and say that G is of type $L_2(p)$ if each of its composition factors is either a p-group, a p'-group, or isomorphic to $PSL_2(p)$.

Assume that $|P| = p$, so that t is the number of conjugacy classes of elements of order p. Suppose that $C = P \times Z$. Then [5] implies the following: for any p-block B of positive defect there is a sign $\delta_B = \pm 1$ such that the degree of all the exceptional characters in B is congruent to $\delta_B e \pmod{p}$. The degrees of the nonexceptional characters in B are congruent to $\pm 1 \pmod{p}$. Further, there is a one-to-one correspondence between p-blocks of positive defect and irreducible (linear) characters η of Z: an irreducible character χ is in B_η if and only if $p \nmid \chi(1)$ and $\chi_Z = \chi(1)\eta$. It follows that if χ and ξ are in p-blocks of positive defect, then all the irreducible constituents of $\chi\xi$ that are not of defect zero lie in a unique p-block, which we denote by $B(\chi\xi)$. B_1 will mean the principal p-block.

THEOREM 1. *Assume that $G = G'$ is not of type $L_2(p)$, $C = P \times Z$, $|P| = p$ and $t \geqslant 3$. Let χ, ξ be exceptional characters of G in p-blocks of positive defect,*

with either $\chi(1) = np + e$ *and* $\xi(1) = mp + e$ *or* $\chi(1) = np - e$ *and* $\xi(1) = mp - e$ *for some positive integers n, m. Let* $B = B(\chi\xi)$. *If* $\delta_B = -1$, *then there is a function* $f(n, m)$, *independent of p, such that* $e \leqslant f(n, m)$. *If* $\delta_B = +1$, *then there is a function* $f(n, m, t)$ *(depending on n, m, and t but not on p) such that* $e \leqslant f(n, m, t)$.

Note that [8, I.17.13] and [6, Theorem 1] imply that when $\chi(1) \equiv \xi(1) \equiv e$ (mod p), the assumption that G is not of type $L_2(p)$ forces m and n to be positive.

If $\chi(1) = p - e = \xi(1)$, then if $\delta_B = -1$ we must have $e = 2$ (and if χ or ξ is faithful, the assumption $C = P \times Z$ is redundant) [7, (4.1), (2.1)], [3, Theorem 1 and Remark (iii)]. If $\delta_B = +1$ then $e \leqslant t - 3$ [2], [3, Theorem 1 and Remark (iv)]. Theorem 1 may be considered a generalization of these results, but the upper bounds obtained for e, namely

$$f(n, m) = 4n + 4m + 16nm, \quad f(n, m, t) = n + m + nm(t + 1)$$

seem far from sharp. For instance, they do not quite reach the above-mentioned bounds when $m = n = 1$. Furthermore, there are no examples known where the hypotheses of Theorem 1 are satisfied with $\chi(1) = p - e = \xi(1)$, $\delta_B = +1$, and $e > 2$. However, we obtain a nearly best possible result in

THEOREM 2. *Assume that* $G = G'$, $|P| = p$, *and that G has an irreducible character* $\chi \in B_1$ *with* $(3p + 1)/2 < \chi(1) < 2p - 1$. *Then* $\chi(1)$ *equals* $2p - 2$ *or* $2p - 4$.

The case $\chi(1) = 2p - 2$ occurs. For example, $PSL_2(25)$ has a faithful character of degree 24 in the principal 13-block. However, I do not know if there is an instance of the case $\chi(1) = 2p - 4$ under the hypotheses of Theorem 2. The theorem is almost a "translation p units to the right" of a special case of [7, Theorem 1], namely, that if $(p + 1)/2 < \chi(1) < p - 1$, and the other hypotheses of Theorem 2 hold, then $\chi(1) = p - 2$. Theorem 2 contains no assumption on C, but note (3.1) of the proof.

Analogs of our theorems may hold without the assumptions $C = P \times Z$ or $\chi \in B_1$. However, their proofs would involve extensions of the available machinery which I have not yet completed. Theorems 1 and 2 are proved below with the techniques of [7], [1], and [3].

2. Proof of Theorem 1.

Notation. R is the ring of integers in a p-adic number field F such that F and $R/J(R)$ are splitting fields for all subgroups of G. R' is a finite extension of R with ramification index at least 2 [8, p. 103]. $K = R'/J(R')$. Y^* is the dual of a KG-module Y, ζ^* the complex conjugate of a complex character ζ. Since $C = P \times Z$ implies N/P is abelian, the indecomposable KN-modules are described in [1, §2]. We freely use the notation and terminology of [1]. In

particular, if $N = PH$ with H a p'-group, then *char H* is the set of all linear characters: $H \to K$. The particular character $\alpha \in \text{char } H$ is defined by

$$h^{-1}yh = y^{\alpha(h)}$$

for all $h \in H$, where $\langle y \rangle = P$. A typical indecomposable *KN*-module is $V_m(\lambda)$, $1 \leqslant m \leqslant p$, $\lambda \in \text{char } H$. If X is a *KG*-module with $X_N = \bigoplus \Sigma_i V_{d_i}(\lambda_i)$, the *H-values* of X are all the $\lambda_i \alpha^{-j}$, $0 \leqslant j \leqslant d_i - 1$ [1, Proposition 2.11]. The *main values* are the λ_i. Such a λ_i is a *projective main value (pmv)* of X if $d_i = p$, a *nonprojective main value (npmv)* if not (a given character could be both). If X is nonprojective and indecomposable, $X \leftrightarrow V_m(\lambda)$ means that $V_m(\lambda)$ is the Green correspondent of X (so $V_m(\lambda) | X_N$ and λ is the unique npmv of X) [8, III.5.6].

In addition to all the hypotheses of Theorem 1, we may assume $e > 2$. Let L', M' be R'-free, $R'G$-modules affording χ, ξ resp. such that $L = L'/J(R')L'$ and $M = M'/J(R')M'$ are indecomposable [8, I.17.13]. Let $\nu = \pm 1$ such that $\chi(1) \equiv \xi(1) \equiv \nu e \pmod{p}$. The Green correspondence implies that when $\nu = 1$,

$$L_N = V_e(\lambda) \oplus \sum_{\sigma \in \mathbb{S}} V_p(\sigma), \qquad M_N = V_e(\mu) \oplus \sum_{\tau \in \mathbb{T}} V_p(\tau)$$

for some λ, $\mu \in \text{char } H$, where $\mathbb{S}$, $\mathbb{T}$ are subsets of char H, counting possible multiplicities, with $|\mathbb{S}| = n$, $|\mathbb{T}| = m$. If $\nu = -1$,

$$L_N = V_{p-e}(\lambda) \oplus \sum_{\sigma \in \mathbb{S}} V_p(\sigma), \qquad M_N = V_{p-e}(\mu) \oplus \sum_{\tau \in \mathbb{T}} V_p(\tau)$$

where $|\mathbb{S}| = n - 1$, $|\mathbb{T}| = m - 1$. Then [1, Lemmas 2.4, 2.5] (and the fact $\alpha^e = 1$) imply that when $\nu = 1$,

$$(L \otimes_K M)_N = \bigoplus \sum_{i=0}^{e-1} V_{2i+1}(\lambda\mu\alpha^{1+i}) \oplus \sum_{\sigma \in \mathbb{S}} \sum_{i=0}^{e-1} V_p(\mu\sigma\alpha^{-i})$$

$$\oplus \sum_{\tau \in \mathbb{T}} \sum_{i=0}^{e-1} V_p(\lambda\tau\alpha^{-i}) \oplus \sum_{\mathbb{S}, \mathbb{T}} \sum_{i=0}^{p-1} V_p(\sigma\tau\alpha^{-i}). \qquad (2.1)$$

When $\nu = -1$, [1, Lemmas 2.5, 2.6] yields

$$(L \otimes_K M)_N = \bigoplus \sum_{i=0}^{e-1} V_{2i+1}(\lambda\mu\alpha^i) \oplus \sum_{i=0}^{p-2e-1} V_p(\lambda\mu\alpha^{-i})$$

$$\oplus \sum_{\sigma \in \mathbb{S}} \sum_{i=0}^{p-e-1} V_p(\mu\sigma\alpha^{-i})$$

$$\oplus \sum_{\tau \in \mathbb{T}} \sum_{i=0}^{p-e-1} V_p(\lambda\tau\alpha^{-i}) \oplus \sum_{\mathbb{S}, \mathbb{T}} \sum_{i=0}^{p-1} V_p(\sigma\tau\alpha^{-i}). \qquad (2.2)$$

Let $L_i \leftrightarrow V_{2i+1}(\lambda\mu\alpha^{1+i})$ when $\nu = 1$, $L_i \leftrightarrow V_{2i+1}(\lambda\mu\alpha^i)$ when $\nu = -1$. The

Green correspondence implies

$$L \otimes M = \oplus \sum_{i=0}^{e-1} L_i \oplus Q$$

where Q is projective.

Now since $|\langle \alpha \rangle| = e$, any $\gamma \in$ char H can occur at most $[u/e] + 1$ times as a pmv of $\oplus \Sigma_{i=0}^u V_p(\rho \alpha^{-i})$ (where ρ is fixed in char H and square brackets denote the greatest integer symbol). Then (2.1) and (2.2) imply

(2.3) *Any* $\gamma \in$ char H *can occur as a pmv of* $L \otimes M$ *at most* $n + m + nm(t + 1)$ *times if* $v = 1$, *and at most* $t - 1 + (n - 1)t + (m - 1)t + (n - 1)(m - 1)(t + 1) = nm(t + 1) - n - m$ *times if* $v = -1$.

For each i with $0 \leqslant i \leqslant e/2 - 1$ (e even), or $0 \leqslant i \leqslant (e - 3)/2$ (e odd), there is an R'-free, $R'G$-module X_i such that

$$X_i/J(R')X_i \approx L_i \oplus L_{e-1-i} \quad [3, \text{Lemma } 2.1].$$

Also, there is an R'-free, $R'G$-module Y with $Y/J(R')Y \approx L_{e/2}$ (e even) or $Y/J(R')Y \approx L_{(e-1)/2}$ (e odd). Now the exceptional characters in B occur in the character afforded by each X_i with total multiplicity at least 2 if $\delta_B = 1$, and at least $t - 2$ if $\delta_B = -1$. They occur in the character afforded by Y with total multiplicity at least 1 if $\delta_B = 1$, and at least $t - 1$ if $\delta_B = -1$ [3, Lemma 2.2].

Let U be an irreducible KG-module which is a modular constituent of an exceptional character in B. Then U is a constituent of $L_i \oplus L_{e-1-i}$, $L_{e/2}$ or $L_{(e-1)/2}$ with multiplicity at least each of the above, resp. So if $\delta_B = +1$, then U occurs in $\oplus \Sigma_{i=0}^{e-1} L_i$ at least $2(e/2) = e$ times if e is even, and at least $2(e - 1)/2 + 1 = e$ times if e is odd. If $\delta_B = -1$, then U is a constituent of $\oplus \Sigma_{i=0}^{e-1} L_i$ at least $(t - 2)(e/2 - 1) + (t - 1) = e(t - 2)/2 + 1$ times if e is even, and at least $(t - 2)(e - 1)/2 + (t - 1) = e(t - 2)/2 + t/2$ times if e is odd.

Suppose $\delta_B = 1$. By the argument of [3, p. 552] (where it suffices that $e \leqslant (p - 1)/3$), there exists a module U, as above, with a pmv π. Thus π occurs at least e times as a pmv of $L \otimes M$. Now (2.3) implies $e \leqslant n + m + nm(t + 1)$. Theorem 1 is proved in this case, with

$$f(n, m, t) = n + m + nm(t + 1).$$

Suppose $\delta_B = -1$. The argument of [3, p. 553] shows that there exists a modular constituent U (with npmv β, say) of the exceptional characters in B such that if there is an irreducible KG-module W which absorbs β (see [3, p. 546]), then W has a pmv γ. If W does not exist, or if $\gamma = \beta$, then [3, Lemma 2.7] implies that β is a pmv of $L \otimes M$ at least $e(t - 2)/2$ times. Then by (2.3),

$$e(t - 2)/2 \leqslant n + m + nm(t + 1)$$

whence

$$e \leqslant (n + m + nm(t + 1))(2/(t - 2)) \leqslant 2n + 2m + 8nm$$

since $t \geqslant 3$. If W exists with $\gamma \neq \beta$, then [3, Lemma 2.7] yields that γ, β occur together as pmv's of $L \otimes M$ at least $e(t - 2)/2$ times. Then (2.3) yields

$$e(t - 2)/2 \leqslant 2(n + m + nm(t + 1)),$$

so that

$$e \leqslant (n + m + nm(t + 1))(4/(t - 2)) \leqslant 4n + 4m + 16nm.$$

We take $f(n, m) = 16nm + 4n + 4m$ to complete the proof of Theorem 1.

REMARK. The bounds may be sharpened by handling the small values of t, or the cases $\nu = \pm 1$, separately. They may also be improved if we know that λ, μ, and the linear characters in $\mathcal{S}$ and $\mathcal{T}$ are not all identical (they usually are not).

3. Proof of Theorem 2. Assume that G and χ satisfy the hypotheses of Theorem 2. By [5, Theorem 11], the degrees of the characters in B_1 are congruent to ± 1 or $\pm e \pmod{p}$. Then $(3p + 1)/2 < \chi(1) < 2p - 1$ implies $\chi(1) = 2p - e$ where $1 < e < (p - 1)/2$, so that χ is an exceptional character in B_1.

(3.1) *It suffices to assume χ is faithful, $C = P$ and char $H = \langle \alpha \rangle$.*

PROOF. Let J be the kernel of χ. If $J \supseteq P$, then χ, one of $t \geqslant 3$ p-conjugate characters, is invariant under p-conjugation, a contradiction. So J is a p'-group and $B_1(G/J) = B_1(G)$ [8, IV.4.12, V.4.3]. Therefore, $\chi \in B_1(G/J)$, and the hypotheses of Theorem 2 are satisfied for group G/J and character χ. So we may assume $J = \langle 1 \rangle$. Then $C = P$ [8, IV.4.21]. So N/P is cyclic and char $H = \langle \alpha \rangle$.
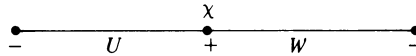
(3.2) *G is simple.*

PROOF. $O_{p'}(G)$ is the kernel of B_1 [8, IV.4.12], so χ faithful implies $O_{p'}(G) = \langle 1 \rangle$. Thus if J is a proper normal subgroup of G, then $P \subseteq J$. The Frattini argument implies $G = JN$. Then $G/J \approx N/N \cap J$ is a homomorphic image of N/P, and hence is cyclic. Then $G' \subseteq J$ yields $J = G$.

(3.3) *The assumptions of Theorem 1 hold with $\chi = \xi$, $L = M$, $\chi(1) = 2p - e$, $B = B_1$, and $\delta_B = -1$.*

PROOF. Since $e < (p - 1)/2$, $G \not\approx PSL_2(p)$. Then (3.2) says G is not of type $L_2(p)$. The rest is clear.

(3.4) *If L is reducible it has exactly two constituents, say U and W, where*

dim $U = p - s$, dim $W = p - e + s$, *some $s < e$. Also, U, W are both self-dual, and the node for χ (and the other exceptionals) in the Brauer graph of B_1 appears as*

$$\underset{-}{\bullet} \underline{\qquad\qquad} \underset{U}{} \underline{\qquad} \overset{\overset{\chi}{\bullet}}{\underset{+}{}} \underline{\qquad} \underset{W}{} \underline{\qquad\qquad} \underset{-}{\bullet}$$

Furthermore, if $U_N = V_{p-s}(\beta)$ then $W_N = V_{p-e+s}(\beta\alpha^s)$.

PROOF. $\chi(1) \equiv -e \pmod{p}$ implies that each modular constituent U of L has

$$\operatorname{sep} U = -\operatorname{rem} U$$

(see [1, §4] and [9]). Thus dim $U \geqslant p - e$. So $\chi(1) = 2p - e$ forces L to have at most two (necessarily distinct) irreducible constituents, say U and W. Suppose there are two. Then sep U + sep $W = e$, so dim $U = p - s$, dim $W = p - e + s$, some $1 \leqslant s \leqslant e - 1$. χ lies on the real stem of B_1, and symmetry about the stem means that U and W are self-dual. Let $U_N = V_{p-s}(\beta)$, $W_N = V_{p-e+s}(\tau)$. We may assume there is an exact sequence $0 \to U \to L \to W \to 0$. Then [1, Proposition 4.5] implies $\tau = \beta\alpha^{p-e+s-1} = \beta\alpha^s$.

(3.5) *If e is even then L is irreducible and $L_N = V_{p-e}(-1) \oplus V_p(1)$, where $-1 = \alpha^{e/2}$.*

PROOF. Suppose L is reducible. Let U, W be as in (3.4) and $d = p - s$. $U \approx U^*$ implies $\beta^2 = \alpha^{d-1}$ [1, Lemma 2.3]. Now $\beta \in \langle\alpha\rangle$, so e even implies d is odd. $G = G'$ means that every representation of G is unimodular. Then [1, Lemma 2.3] yields that $\beta^d = \alpha^{d(d-1)/2}$. Since d is odd, β may be found from β^2 and β^d: namely, $\beta = \alpha^{(d-1)/2}$. A similar argument for W implies

$$\beta\alpha^s = \alpha^{(p-e+s-1)/2} = \alpha^{(d-1)/2}\alpha^s\alpha^{e/2}.$$

Hence, $\beta = \alpha^{(d-1)/2}\alpha^{e/2}$ so that $\alpha^{e/2} = 1$, a contradiction.

So L is irreducible and must lie on the real stem. Let $L_N = V_{p-e}(\lambda) \oplus V_p(\sigma)$. Then $L \approx L^*$ implies $\lambda^2 = \alpha^{p-e-1} = 1$. If $\lambda = 1$ then $L \approx 1_0$ (the trivial one-dimensional KG-module) [1, Corollary 4.7], a contradiction. So $\lambda = -1$. Again, $G = G'$ and [1, Lemma 2.3] imply

$$1 = \lambda^{p-e}\alpha^{-(p-e)(p-e-1)/2}\sigma^p\alpha^{-p(p-1)/2} = \sigma^p = \sigma.$$

(3.6) *Assume e is even. Let $S \oplus T$ denote the decomposition of $L \otimes L$ into symmetric and skew parts, respectively. Define $\gamma \in \langle\alpha\rangle$ to be even if $\gamma \in \langle\alpha^2\rangle$, odd if not. Then*

$$S_N = \bigoplus \sum_{0 < i < e-1; i \equiv 0(\text{mod } 2)} V_{2i+1}(\alpha^i) \oplus (3t-3) \sum_{\gamma \text{ even}; \gamma \neq \pm 1} V_p(\gamma)$$

$$\oplus (t-1) \sum_{\gamma \text{ odd}; \gamma \neq -1} V_p(\gamma) \oplus (3t-1)V_p(1) \oplus x V_p(-1),$$

$$T_N = \bigoplus \sum_{0 < i < e-1; i \equiv 1(\text{mod } 2)} V_{2i+1}(\alpha^i) \oplus (t-1) \sum_{\gamma \text{ even}; \gamma \neq \pm 1} V_p(\gamma)$$

$$\oplus (3t-3) \sum_{\gamma \text{ odd}; \gamma \neq -1} V_p(\gamma) \oplus (t-1)V_p(1) \oplus (4t-2-x)V_p(-1)$$

where $x = 3t - 2$ if $\alpha^{e/2}$ is even, and $x = t$ if $\alpha^{e/2}$ is odd.

PROOF. S and T are KG-submodules of $L \otimes L$, and S_N, T_N are the symmetric (resp. skew) summands of $(L \otimes L)_N$. S has K-dimension $(2p - e)(2p - e + 1)/2$ and
$$\dim T = (2p - e)(2p - e - 1)/2$$
(see [1, §3]). The H-values of L_N are $\{\alpha^{(e/2)-i} | 0 < i < p - e - 1\} \cup \{\alpha^{-i} | 0 < i < p - 1\}$ by (3.5) and [1, Proposition 2.11]. Squaring each element of this set yields every even character $4t - 2$ times, plus the trivial character twice more. Then [1, (3.1)] implies

(3.7) *The H-values of S_N consist of the H-values of T_N, together with each nontrivial even character of $\langle \alpha \rangle$, with additional multiplicity $4t - 2$, and the trivial character, with additional multiplicity $4t$.*

Now by (2.2),

$$(L \otimes L)_N = \bigoplus \sum_{i=0}^{e-1} V_{2i+1}(\alpha^i) \oplus \sum_{i=0}^{p-2e-1} V_p(\alpha^{-i})$$

$$\oplus 2 \sum_{i=0}^{p-e-1} V_p(\alpha^{(e/2)-i}) \oplus \sum_{i=0}^{p-1} V_p(\alpha^{-i})$$

$$= \bigoplus \sum_{i=0}^{e-1} V_{2i+1}(\alpha^i) \oplus (4t-4) \sum_{\gamma \neq \pm 1} V_p(\gamma)$$

$$\oplus (4t-2)V_p(1) \oplus (4t-2)V_p(-1). \tag{3.8}$$

The Krull-Schmidt theorem implies the summands in (3.8) are distributed between S_N and T_N. Since $V_{p-e}(-1)|L_N$, [1, (3.1)] implies the symmetric (resp. skew) part of $V_{p-e}(-1) \otimes V_{p-e}(-1)$ is a summand of S_N (resp. T_N). So by [1, Lemma 3.3],

$$\bigoplus \sum_{0 < i < e-1, i \equiv 0(\text{mod } 2)} V_{2i+1}(\alpha^i) \bigg| S_N, \quad \bigoplus \sum_{0 < i < e-1; i \equiv 1(\text{mod } 2)} V_{2i+1}(\alpha^i) \bigg| T_N.$$

$$\tag{3.9}$$

Then a dimension argument implies S_N must have exactly two more projective indecomposable summands than T_N does.

Now (3.9) shows that from the nonprojective summands alone, each odd γ occurs as an H-value of T_N twice more than in S_N, and the even characters are balanced as H-values. Neglecting pmv's, the projective summands must contribute $2t$ more of each character as an H-value of S_N than as an H-value of T_N. So without the pmv's, S_N has $2t$ more of each even γ, and $2t - 2$ of each odd γ, occurring as H-values than T_N does. Now (3.7) implies for each even $\gamma \neq 1$, $V_p(\gamma)$ must occur in S_N $2t - 2$ more times than in T_N. Also, for each odd γ, $V_p(\gamma)$ is a summand of T_N $2t - 2$ more times than in S_N. Finally, $V_p(1)$ occurs in S_N $2t$ more times than in T_N. The result follows from (3.8).

(3.10) *If e is even then e equals 2 or 4.*

PROOF. Suppose $e \geqslant 6$. Let $L_i \leftrightarrow V_{2i+1}(\alpha^i)$, $0 \leqslant i \leqslant e - 1$. Then by the proof of Theorem 1 (with $L = M$), L, the unique modular constituent of the exceptional characters in B_1, is a constituent of $L_0 \oplus L_{e-1}$ with multiplicity at least $t - 2$. Since $L_0 = 1_0$ (by the Green correspondence), L is a constituent of L_{e-1} at least $t - 2$ times.

Because $e < 2e - 5 < p - e$, [1, Lemma 2.6] implies the npmv's of $V_{p-e}(-1) \otimes V_{2e-5}(\alpha^{e-3})$ are

$$\alpha^{e/2}\alpha^{e-3}\alpha^{-(2e-5)+1+i} = \alpha^{(e/2)+3+i}, \qquad 0 \leqslant i \leqslant e - 1.$$

Hence, 1 is a npmv of $L \otimes L_{e-3} \approx L^* \otimes L_{e-3} \approx \mathrm{Hom}_K(L, L_{e-3})$. By [1, Theorem 4.1], there is a nonzero KG-homomorphism from irreducible L into indecomposable $L_{e-3} \not\approx L$. Since L and L_{e-3} are both self-dual, [1, Proposition 4.11] shows that L is a constituent of L_{e-3} with multiplicity at least 2.

Now $L_{e-1} \oplus L_{e-3} | T$ by (3.6) and the Green correspondence. Hence, L is a constituent of T with multiplicity at least t, so that 1 (the pmv of L), is a pmv of T with multiplicity at least t. But (3.6) also implies 1 has multiplicity exactly $t - 1$ as a pmv of T, a contradiction.

(3.11) *If e is odd then $e = 3, 9,$ or 15, L is reducible, and for $U_N = V_{p-s}(\beta)$ as in (3.4), we may assume $s = 2e/3$ and $\beta = \alpha^s$.*

PROOF. Suppose L is irreducible. $L \approx L^*$ implies $V_{p-e}(\lambda) \approx V_{p-e}(\lambda)^*$, hence $\lambda^2 = \alpha^{p-e-1} = 1$ [1, Lemma 2.3]. Then e odd yields $\lambda = 1$. Thus $L \approx 1_0$ [1, Corollary 4.7], a contradiction.

Let U, W be as in (3.4). Since $\chi(1)$ is odd, we may assume s is even (if not, $e - s$ is). Then $U \approx U^*$ implies $\beta^2 = \alpha^{p-s-1}$, whence $\beta = \alpha^{-s/2}$. Now $|H| = e$. Since H is a p'-group, we may identify $\alpha(H)$ with the complex eth roots of unity. If ϕ, ψ are the Brauer characters afforded by U, W resp., then

$$\chi_H = \phi_H + \psi_H = \sum_{i=0}^{p-s-1} \alpha^{-(s/2)-i} + \sum_{i=0}^{p-e+s-1} \alpha^{(s/2)-i}$$

$$= \alpha^{-s/2} + \alpha^{s/2} + (2t-1)\rho$$

where ρ is the character of the regular representation of H.

Since e is odd, there exists an automorphism σ of the field of algebraic numbers which sends a primitive eth root of unity to its square, and fixes the pth roots of 1. Exceptional χ must be fixed by σ, hence

$$\alpha^{-s} + \alpha^s + (2t-1)\rho = (\chi_H)^\sigma = \chi_H = \alpha^{-s/2} + \alpha^{s/2} + (2t-1)\rho$$

and so $\alpha^{-s} + \alpha^s = \alpha^{-s/2} + \alpha^{s/2}$. Then $s \equiv -s/2 \pmod{e}$ implies $3s \equiv 0 \pmod{e}$, whence $3|e$ and $e/3|s$. So s even means $s = 2e/3$. By [3, Theorem 1] with $s = u$, we have $4e/3 = 2s \leqslant e + 5$, whence $e \leqslant 15$. Therefore, $e = 3, 9$, or 15.

(3.12) *The case $e = 15$ is impossible.*

PROOF. By [3, (3.9)] with $s = u = 10$, we have

$$2 = 9 - [15/2] \leqslant t/(t-2),$$

whence $t = 4$. Now, [4, Theorem 2] with $x = 5, s = 10$ implies

$$5 = x \leqslant e - 2s + 2t = 15 - 20 + 8 = 3,$$

a contradiction. (The methods of proof of Theorem 1 will also eliminate this case, without appealing to [4].)

(3.13) *The case $e = 3$ is impossible.*

PROOF. If $e = 3$, then by (3.11), $U_N = V_{p-2}(\alpha^{-1})$. Now [1, Lemma 3.3] implies $L_1 (\leftrightarrow V_3(\alpha))$ is a skew summand of $U \otimes U$, hence

$$\dim L_1 \leqslant (p-2)(p-3)/2 = ((p-5)/2)p + 3.$$

But [3, Lemmas 2.1 and 2.2] show that L_1 lifts to an R'-free, $R'G$-module whose character contains exceptional characters in B_1 with total multiplicity at least $t - 1$. Hence

$$(2t-3)p + 4 = (t-1)(2p-3) \leqslant \dim L_1 \leqslant ((p-5)/2)p + 3,$$

so that $2t - 3 \leqslant (p-7)/2$ and $(p-1)/3 = t \leqslant (p-1)/4$, a contradiction.

(3.14) *The case $e = 9$ cannot occur.*

PROOF. If $e = 9$ then $U_N = V_{p-6}(\alpha^6)$, $W_N = V_{p-3}(\alpha^3)$. In the symmetric summand of $U \otimes U$, α^j is a pmv $t/2$ times if $j \equiv \pm 3$ or $\pm 1 \pmod{e}$, $t/2 - 1$ times otherwise. Furthermore (if $L_i \leftrightarrow V_{2i+1}(\alpha^i)$), $L_0 = 1_0$, L_2 and L_4 are symmetric summands of $U \otimes U$ [1, Lemma 3.3].

As in the proof of Theorem 1, each of U and W is a constituent of L_4 at least $t - 1$ times [3, Lemmas 2.1, 2.2]. There exists an irreducible KG-module M which absorbs α^6 (the npmv of U). Furthermore, U and M are both incident to a common (nonexceptional) node in the graph of B_1, sep $M = $ rem $M \leqslant 5$, and the npmv of M^* is α^2 (see Proposition 2.6 and the proof of Proposition 2.3 in [3]). Thus $M \leftrightarrow V_m(\alpha^{m-3})$, $m = $ rem $M \leqslant 5$ [1, Lemma 2.3].

Since α^6 is a main value (and only a pmv) of the symmetric part of $U \otimes U$ exactly $t/2$ times, [3, Lemma 2.7] implies M is a constituent of L_4 with multiplicity at least $t/2 - 1$. Now dim $M > p$ [1, Theorem 5.7], so M has a pmv γ. Also, α^6 and γ together occur as pmv's of L_4 at least $t - 1$ times [3, Lemma 2.7], so $\gamma \neq \alpha^6$.

Since $V_m(\alpha^{m-3}) \otimes V_5(\alpha^2)$, and hence $M \otimes L_2$, has 1 as a npmv [1, Lemma 2.4], it follows by [1, Theorem 4.1] that

$$\text{Hom}_{KG}(M^*, L_2) \neq 0.$$

Therefore, M (and M^*, if $M \not\approx M^*$) is a constituent of $L_2 \approx L_2^*$. So $t/2 \leqslant$ multiplicity of M in $L_2 \oplus L_4 \leqslant$ multiplicity of γ as a pmv of $L_2 \oplus L_4 \leqslant t/2$. Hence, M (and M^*, if $M \not\approx M^*$) is a constituent of $L_2 \oplus L_4$ with multiplicity exactly $t/2$. Then $2V_p(\gamma) \mid M$, and $2V_p(\gamma) \mid M \oplus M^*$ if $M^* \not\approx M$. Also, α^6 is a pmv of L_4 exactly $t/2$ times, due to the presence of U as a constituent, and U must occur exactly $t - 1$ times in L_4.

Since $V_p(\gamma)^* = V_p(\gamma^{-1})$, γ^{-1} occurs as a pmv of L_2 whenever γ does. So if $\gamma = \alpha^3$, then $\gamma^{-1} = \alpha^6$ is a pmv of $L_2 \oplus L_4$ at least $t/2 + 1$ times, a contradiction. Therefore, $\gamma = \alpha^{\pm 1}$.
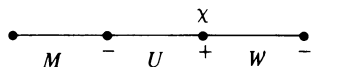
Suppose $M \not\approx M^*$. We have the configuration
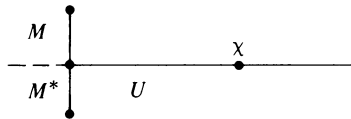
in the Brauer graph, so $m \leqslant 2$. $M_N^* = V_m(\alpha^2) \oplus V_p(\alpha^{\pm 1})$. Since $G = G'$, [1, Lemma 2.3] forces

$$1 = (\alpha^2\alpha)(\alpha^{\pm p}\alpha^{-p(p-1)/2}) = \alpha^{3\pm 1}$$

if $m = 2$, or $1 = \alpha^{2\pm 1}$ if $m = 1$, in either case a contradiction. Therefore, $M \approx M^*$. Then $\alpha^{m-3} = \alpha^2$ implies $m = 5$, whence $M \approx L_2$ and $M_N = V_5(\alpha^2) \oplus V_p(\alpha) \oplus V_p(\alpha^{-1})$. The Brauer graph now contains

There is also an irreducible KG-module X which absorbs α^3 (the npmv of W). X and W are both incident to a common (nonexceptional) node in the Brauer graph, sep $X = \text{rem } X \leqslant 2$, and the npmv of X^* is α^5. X has a pmv, say π. Since α^6 is a pmv of $L_4 \approx L_4^*$ exactly $t/2$ times, so is α^3. Then X is a constituent of L_4 at least $t/2 - 1$ times. Hence, $\pi \neq \alpha^{\pm 3}$, $\alpha^{\pm 1}$, as these pmv's have been produced as much as possible by U, M and W. So X has multiplicity exactly $t/2 - 1$ as a constituent of L_4, hence [3, Lemma 2.7] implies W has multiplicity exactly $t - 1$.

$X \leftrightarrow V_x(\alpha^{x+3})$, $x \leqslant 2$. Suppose $X \not\approx X^*$. We have the picture
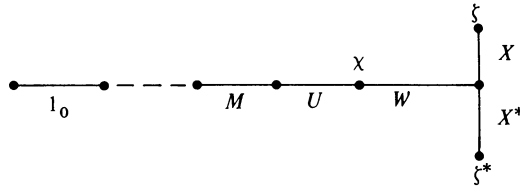
where sep $W = 3$ implies $x = 1$.

Let $X_N = V_1(\alpha^4) \oplus \Sigma_i V_p(\pi_i)$, $\pi_i \neq \alpha^{\pm 1}$, $\alpha^{\pm 3}$. Then

$$X_N^* = V_1(\alpha^5) \oplus \Sigma_i V_p(\pi_i^{-1}).$$

If we identify $\alpha(H)$ with the complex 9th roots of unity, then $\zeta_H = n\rho_H + \alpha^4 + \Sigma\pi_i$, $n = t(\zeta(1) - 1)/p$. Thus the irreducible constituents of ζ_H determine the main values of X.

Let σ be an automorphism of the field of algebraic numbers which squares a primitive 9th root of 1. Then $\zeta_H^\sigma = n\rho_H + \alpha^{-1} + \Sigma\pi_i^2$. So if $\zeta^\sigma = \zeta$ or ζ^*, then α^{-1} is a pmv of X or X^*, a contradiction. Now $(\zeta^{\sigma^2})_H = n\rho_H + \alpha^7 + \Sigma\pi_i^4$. If $\zeta^{\sigma^2} = \zeta^*$, then $\zeta^{\sigma^4} = \zeta$ implies $\zeta^{\sigma^2} = \zeta$ since σ has order 6. But if $\zeta^{\sigma^2} = \zeta$, then

$$\zeta_H = n\rho_H + \alpha^7 + \Sigma\pi_i^4 = n\rho_H + (\alpha^7)^4 + \Sigma\pi_i^{16}.$$

Hence, $\alpha^{28} = \alpha$ is a pmv of X, a contradiction. It follows that ζ, ζ^σ, ζ^{σ^2}, ζ^*, $(\zeta^*)^\sigma$, $(\zeta^*)^{\sigma^2}$ are six distinct characters in B_1, none of which lies on the real stem. The above configuration, with sep $M = 5$, shows this is impossible.

We may assume that $X \approx X^*$. Then $X \leftrightarrow V_2(\alpha^5)$ and the graph now contains
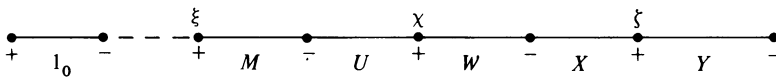
where $Y \leftrightarrow V_{p-1}(\alpha^4)$.

Suppose $\dim X = p + 2$ and $\dim Y = p - 1$. Then $V_p(\pi) \approx V_p(\pi)^*$

implies $\pi = 1$. By [1, Lemma 2.2]

$$\zeta_H = \alpha^5 + \alpha^4 + \alpha^0 + 2t\rho_H.$$

Thus, ζ has two algebraic conjugates in B_1, which must also lie on positive nodes on the real stem. But since $\zeta(1) = 2p + 1$, $\chi(1) = 2p - 9$ and $\xi(1) \geqslant 3p + 1$, there are not enough nodes available. Hence, we may assume $\zeta(1) \geqslant 3p + 1$.

Let L_4' be an R'-free, $R'G$-module with $L_4'/J(R')L_4' = L_4$. Since the multiplicity of U and W in L_4 is exactly $t - 1$, [3, Lemma 2.2] implies the exceptional characters must occur in the character afforded by L_4' with total multiplicity exactly $t - 1$. Then in order that M and X occur in L_4 $t/2 - 1$ times, each of ξ and ζ has multiplicity $t/2 - 1$ in the character of L_4'. If A denotes the symmetric part of $U \otimes U$, we have

$$\begin{aligned} p(5t - 9) + t + 8 &= (t/2 - 1)(6p + 2) + (t - 1)(2p - 9) \\ &\leqslant (t/2 - 1)(\xi(1) + \zeta(1)) + (t - 1)\chi(1) \\ &\leqslant \dim L_4 \leqslant \dim A - \dim L_2 - 1 \\ &= (p - 6)(p - 5)/2 - (2p + 5) - 1 \\ &= p(p - 15)/2 + 9, \end{aligned}$$

so that $5t - 9 < (p - 15)/2$. Then $(p - 1)/9 = t < (p + 3)/10$ and $p < 37$. But $e = 9$ and $t \geqslant 4$ imply $p \geqslant 37$, a final contradiction.

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